Exercise 4

1. Let U be an open subset of $\mathbb{C}$ and let A be a discrete subset of U. Let c be a closed curve in $U - A$ which is homotopic to a constant as a curve in U. Show that the set of elements $a \in A$ such that the index $I(c, a)$ is different from zero is finite.

2. Let $c_1, c_2 : [0,1] \to \mathbb{C}$ be the picewise C^1 closed paths such that $|c_1(t)| < |c_2(t)|$ for every $t \in [0,1]$. Let $c : [0,1] \to \mathbb{C}$ given by $c(t) := c_1(t) + c_2(t)$. Show that $I(c, 0) = I(c_2, 0)$.

3. Let U be a connected open subset of $\mathbb{C}$ and let $c : [0,1] \to U$ be a closed curve in U. We say that c is homologous to zero in U if the following holds:
Let $A = \{a \in \mathbb{C} - Image(c) : I(c,a) = 0\}$; then $A \subset U$.

(I) Show that if c is homotopic to a constant in U then c is homologous to zero in U.
(II) Can you show that the converse of (I) is not true in general by giving a counterexample?

4. Let U be an open subset of $\mathbb{C}$, $a \in U$ and let $(h_n)_n$ be a sequence of holomorphic functions defined on U. For every $n \in \mathbb{N}$ we define $f_n = (z-a)h_n(z)$. Show that if the sequence $(f_n)_n$ converges uniformly to zero in U then the sequence $(h_n)_n$ also converges uniformly to zero in U.

5. Let f be an entire fuction and $a \in \mathbb{C}$ a zero of f. Show that:

$$|f(z)| \leq 2|z-a| sup\{|f(w)| : |z-w| = 1\}.$$

6. Let $f \in H(B(0,1))$ with $f(0) = 1$ and $Re(f)(z) \in R_+$ for every $z \in B(0,1)$. We define $h(z) = \frac{f(z)-1}{f(z)+1}$. Show that $|h(z)| \leq |z|$.
(Hint: using e^{-f} first show that $Re(f)(z) > 0$ for every $z \in B(0,1)$.)

7. (I) Let $f : B(0,1) \to B(0,1)$ be a holomorphic function. Assume that $a \neq b$ are elements of $B(0,1)$ and that $f(a) = \alpha$ and $f(b) = \beta$. Show that $|\frac{\beta-\alpha}{1-\bar{\alpha}\beta}| \leq |\frac{a-b}{1-\bar{a}b}|$ and $|f^{(1)}(a)| \leq \frac{1-|\alpha|^2}{1-|a|^2}$. As a result show that $|f(z)| \leq \frac{|f(0)|+|z|}{1+|f(0)||z|}$.
(II) Let f be holomorphic on $B(0,r)$ and assume that $|f(z)| \leq M$ for all z. Show that $|\frac{f(z)-f(w)}{z-w}| \leq \frac{2Mr}{|r^2-\bar{z}w|}$.

8. (I) Let U be a simply connected open subset of $\mathbb{C}$ and let $f \in H(U)$. Suppose that f is nowhere zero on U. Show that for any $n \in \mathbb{Z}, n \neq 0$, there is $h \in H(U)$ such that $h^n = f$.

(II) Let U be a simply connected open subset of $\mathbb{C}$ and suppose that f is a holomorphic function on U that does not assume the values 0 or 1. Show that then there is $g \in H(U)$ such that $f(z) = -exp(i\pi cosh[2g(z)])$ for z in U. $(cosh(z) := \frac{exp(z)+exp(-z)}{2})$

(III) Let f, U, and g be as in part (II). Can you show that $g(U)$ contains no disk of raduis 1?