Exercise 3

1. Consider that all transition functions for an atlas of a vector bundle are identity. Show that the vector bundle is trivial. As a result show that the torus is parallelizable.

2. (I) Let S^k denote the sphere of dim k. Show that S^{2n+1} has a nowhere zero vector field.
   (II) Show that S^3 is parallelizable. Is S^7 parallelizable?

3. Let X be a vector field over a manifold M. For $x \in M$ we recall that an orbit or a trajectory passing through x is the set $\{\varphi_t(x) | t \in (\alpha(x), \beta(x))\}$. Show that the orbits of X are either disjoint or equal. Also show that every trajectory is one of the following type: a point, an embedding of a circle or an injective immersion of a real line.

4. Let M be a connected manifold of dimension at least 2. Let $x, y \in M$. Show that there exists a diffeomorphism f of M sending the point x to y (using a flow of a well chosen vector field, first show the result for the case where x and y are close enough).

5. Let X be vector field over a manifold M. We say X adimitting a first integral if there exists a function $f : M \to \mathbb{R}$ such that $X.f = 0$.
   (I) For such a vector field X show that if f is a proper function then X is complete.
   (II) Newton's equations for a moving of a particle of mass m in a potiential field on $\mathbb{R}^n$ are given by, $\frac{d}{dt^2}\mathbf{q}(t) = -\frac{1}{m}\nabla V(\mathbf{q}(t))$, for $V : \mathbb{R}^n \to \mathbb{R}$ a smooth function. Using (I) show that if there exists $a, b \in \mathbb{R}$, $b \geq 0$ such that $\frac{1}{m}\nabla V(\mathbf{q}) \geq a - b \parallel q \parallel^2$, then every solution exists for all time.

6. Let $X = y^2 \frac{\partial}{\partial x}$ and $Y = x^2 \frac{\partial}{\partial y}$. Show that X and Y are complete on $\mathbb{R}^2$ but $X + Y$ is not.

7. Let M be a smooth manifold and let $f : M \to \mathbb{R}$ be a smooth function. Let J be an open interval of $\mathbb{R}$ such that $J \subset f(M)$ and it contains no critical value of f. Let $a, b \in J$ such that $a < b$.
   (I) Show that there exists a smooth vector field X on M with $supp(X) \subset f^{-1}(J)$ such that for every $x \in f^{-1}([a, b])$ we have $T_x f(X(x)) = \frac{\partial}{\partial x}$.
   (II) Show that there exists a diffeomorphism $g : f^{-1}(]a, b[) \to f^{-1}(a) \times]a, b[$ such that $pr_2 \circ g = f$.

(III) Using II can you show that every smooth function on torus T^2 with value in $\mathbb{R}$ has at least three critical points?

8. Let X, Y be two C^∞ vector fields over a smooth manifold M with local flows φ_t, ψ_t. Let $x \in M$ and let $c(t) := \varphi_t \circ \psi_t \circ \varphi_{-t} \circ \psi_{-t}(x)$. Show that $\frac{d}{dt^2}_{|t=0} c(t) = 2[X, Y](x)$.