Exercise 2

1. Let $R > r > 0$, we define the revolution torus T as the subset of $\mathbb{R}^3$ given by the equation
$$(\sqrt{x^2+y^2}-R)^2+z^2=r^2$$
(I) Show that T is a submanifold of $\mathbb{R}^3$.
(II) Let $R=2$ and $r=1$ and let the function $f : T \to \mathbb{R}$ be given by $f(x,y,z)=x$. Find the critical points of f.
(III) Describe the level sets of f. Which ones are submanifold of $\mathbb{R}^3$?

2. Let's pick up g disc $D((x_i,y_i),r_i)$, $1 \le i \le g$, in $\mathbb{R}^2$ which are disjoint and are contained in $\overline{D(0,R)}$. Let the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by
$$f(x,y)=(R^2-(x^2+y^2))\prod_{i=1}^{g}((x-x_i)^2+(y-y_i)^2-r_i^2)$$
Show that the set $S=\{z^2=f(x,y)\}$ is a compact connected submanifold of $\mathbb{R}^3$. Describe this submanifold.

3. Show that an infinite subgroup of $\mathbb{R}/\mathbb{Z}$ which is generated by one element is dense.
(I) Let a be an irrational number. Show that the map α from $\mathbb{R}$ to $\mathbb{R}^4$ which sends t to $(e^{2\pi it}, e^{2\pi iat})$ is an injective immersion.
(I) What's the closure of $\alpha(\mathbb{R})$? Conclude that $\alpha(\mathbb{R})$ is not a submanifold of $\mathbb{R}^4$.

4. Let P be a non constant real homogeneous polynomial of degree n. Let a be a real number which is not a root of P. Show that $P^{-1}(a)$ is a submanifold of $\mathbb{R}^n$ of dimension $n-1$ and show that these submanifolds are diffeomorphic for $a > 0$.

5. Let F_n denote the set of sequences of nested linear subspaces $V_1 \subset V_2 \subset \cdots \subset V_{n-1}$ in $\mathbb{R}^n$, where $dim V_i = i$. Show that F_n is a compact manifold.

6. Show that any manifold of dim n has an atlas $\{(U_i,\varphi_i)\}$ for which $\varphi_i(U_i)=\mathbb{R}^n$.

7. Let M be a C^k manifold. show that the diagonal $\Delta = \{(m,m)|m \in M\}$ is a closed C^k submanifold of $M \times M$.

8. Show that $SL(n,\mathbb{C}), U(n), SU(n), GL(n,\mathbb{R}), SL(n,\mathbb{R}), O(n)$ and $SO(n)$ are submanifolds of $GL(n,\mathbb{C})$.

(I) Show that $SO(2)$ and $U(1)$ are diffeomorphic to S^1.
(II) Show that $U(n), SU(n), O(n)$ and $SO(n)$ are compact.
(III) Show that $U(n), SU(n), SO(n)$ are connected and $O(n)$ has two connected components.