Exercise 1

1. Show that the Implicit Function Theorem implies the Inverse Function Theorem.

2. By giving an example show that in the Inverse Function Theorem the continuity hypothesis on the derivative cannot be dropped.

3. Let $I : GL(n, \mathbb{R}) \to GL(n, \mathbb{R})$ be given by $A \to A^{-1}$. Show that I is of class C^1 and $DI(A).B = -A^{-1} \circ B \circ A^{-1}$.

4. Let $det : M(n, \mathbb{R}) \to \mathbb{R}$ be given by $A \to det(A)$. Show that det is of C^1 and calculate $Ddet(A)$ the derivative of det at a point $A \in M(n, \mathbb{R})$.

5. Using the Inverse Function Theorem show that simple roots of polynomials are smooth functions of their coefficients. Conclude that simple eigenvalues of linear operators of $\mathbb{R}^n$ are smooth functions of the operator.

6. Let $f : U \subset \mathbb{R}^m \to \mathbb{R}^n$ be a C^1 map.

(I) Show that the set $X_r := \{x \in U | rankDf(x) \geq r\}$ is an open subset of U.

(II) Let a be the maximal rank of $Df(x)$ in U. Show that the set $Y_a := \{x \in U | rankDf(x) = a\}$ is open in U.

(III) We define $Y_i := int\{x \in U | rankDf(x) = i\}$ and let a be the maximal rank of $Df(x)$ in U. Show that $Y_1 \cup \cdots \cup Y_a$ is dense in U.

(IV) Show that if a C^1 map $f : U \subset \mathbb{R}^m \to \mathbb{R}^n$ is injective then $m \leq n$. And if it is surjective to an open subset of $\mathbb{R}^m$ then $m \geq n$.