A natural construction for the real numbers

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We propose a new construction of the real number system, that is built directly upon the additive group of integers and has its roots in the definition due to Henri Poincaré [P, pages 230–233,] of the rotation number of an orientation preserving homeomorphism of the circle. The definitions of addition, multiplication and comparison of real numbers are very natural in our setting. The definition of the multiplication of real numbers does not use the multiplication of integers. The proposed definition of real numbers is illustrated with examples that are irrational, roots of an integral polynomial equation, but not expressible by radicals, or not root of an integral polynomial equation. I thank Sebastian Baader, Etienne Ghys and Domingo Toledo for stimulating discussions. I warmly thank the students in Basel of the first year university classes over many years for their attention and efforts while learning and studying real numbers as explaned in the present paper.

Slopes and definition of the real numbers.

Let $(\mathbf{Z}, +)$ be the set of integers together with the arithmetic operation of addition. The basic objects in our construction are slopes. A *slope* is by definition a map $\lambda : \mathbf{Z} \to \mathbf{Z}$, with the property that the set $\{\lambda(m+n) - \lambda(m) - \lambda(n) \mid m, n \in \mathbf{Z}\}$ is finite. We define two slopes λ, λ' to be *equivalent* if the set $\{\lambda(n) - \lambda'(n) \mid n \in \mathbf{Z}\}$ is finite.

Definition: A *real number* is an equivalence class of slopes.

Let $\mathbf{R}$ denote the set of real numbers.

Basic exemples. For $j \in \mathbf{Z}$, let $\bar{j} : \mathbf{Z} \to \mathbf{Z}$ be the map $\bar{j}(n) := nj$. The map $\bar{j}$ is additive, i.e. satisfies $\bar{j}(n+m) = \bar{j}(n) + \bar{j}(m)$, and hence is a slope for which the expression $\bar{j}(n+m) - \bar{j}(n) - \bar{j}(m)$ takes only the value 0. We identify an integer $j \in \mathbf{Z}$ with the real number represented by the slope $\bar{j}$. After this identification the set of integers $\mathbf{Z}$ becomes a subset of the set of real numbers $\mathbf{R}$. We see that among the real numbers the integers appear as those real numbers, which are representable by an additive slope.

For $p, q \in \mathbf{Z}, q > 0$, let the map $\phi : \mathbf{Z} \to \mathbf{Z}$ be defined by $\phi(n) := \min\{k \in \mathbf{N} \mid qk \geq pn\}, n \in \mathbf{N}$, and by $\phi(-n) = -\phi(n)$ for $n \in \mathbf{Z}, n < 0$. By the definition of $\phi(n)$ we have for $n > 0$ the inequalities

$$q\phi(n) \geq pn, \;\; q(\phi(n) - 1) < pn$$

and for $n < 0$ the inequalities

$$q\phi(n) \le pn,\ q(\phi(n)+1) > pn$$

We conclude for all integers n

$$|q\phi(n) - pn| \le q$$

By adding the above estimates for the integers $n, m, n+m$ we get

$$q|\phi(n+m) - \phi(n) - \phi(m)| = |q\phi(n+m) - q\phi(n) - q\phi(m) - p(n+m) + pn + pm| \le 3q$$

and hence the expression $\phi(n+m) - \phi(n) - \phi(m)$ takes its value in the finite set $\{-3, -2, -1, 0, 1, 2, 3\}$. The map ϕ is a slope. This slope represents the rational number p/q, i.e. the slope ϕ represents a real number which is a solution of the equation $qx = p$. This will become clear, when we have defined multiplication of real numbers in our setting. As for the integers we identify the set of rational numbers $\mathbf{Q}$ with a subset of $\mathbf{R}$. One can characterize the rational numbers as those real numbers, which are representable by a slope λ, such that for some integer $q > 0$ the map $n \in \mathbf{Z} \mapsto \lambda(qn) \in \mathbf{Z}$ is additive.

Let $r \ge 1$ be a rational number. For $n \in \mathbf{N}$ let $L_r(n)$ the greatest $k \in \mathbf{N}$ with $10^{k+1} \le r^n$. The number $L_r(n) + 1$ is the number of digits before the period of the decimal expansion of r^n. We set $L_r(-n) := -L_r(n)$ for negative $n \in \mathbf{Z}$.

Excercise. The map $n \in \mathbf{Z} \mapsto L_r(n) \in \mathbf{Z}$ is a slope and hence defines a real number $[L_r]$ according to our construction. Show $L_{ab} = L_a + L_b$ for rational numbers $a, b \ge 1$. Guess what is $[L_r]$. Define L_r for rational numbers $r > 0$.

Basic operations. We now define the basic arithmetic operations such as *addition* and *multiplication* of real numbers.

Let $a, b \in \mathbf{R}$ be real numbers. Let α, β be slopes representing the real numbers a and b. The map $\alpha + \beta : \mathbf{Z} \to \mathbf{Z}$, which is defined by $(\alpha + \beta)(n) := \alpha(n) + \beta(n)$, is again a slope and its equivalence class is independent of the choice of representatives α, β for a, b. We define the *sum* $a + b \in \mathbf{R}$ of $a, b \in \mathbf{R}$ as the equivalence class of the slope $\alpha + \beta : \mathbf{Z} \to \mathbf{Z}$.

The composition $\alpha \circ \beta : \mathbf{Z} \to \mathbf{Z}$ is again a slope, and we define the *product* $ab \in \mathbf{R}$ as the equivalence class of the composition $\alpha \circ \beta : \mathbf{Z} \to \mathbf{Z}$.

The consistency of this definition follows from the following lemma.

Lemma 1 *Let the slopes α, α' represent $a \in \mathbf{R}$ and the slopes β, β' represent $b \in \mathbf{R}$. Then the compositions $\alpha \circ \beta$ and $\alpha' \circ \beta'$ are equivalent slopes.*

Proof. We first show that the map $\alpha \circ \beta$ is a slope. Let E_α and E_β be finite subsets in $\mathbf{Z}$, such that $\alpha(n+m) - \alpha(n) - \alpha(m) \in E_\alpha$ and $\beta(n+m) - \beta(n) - \beta(m) \in E_\beta$ for $n, m \in \mathbf{Z}$. Hence, for $n, m \in \mathbf{Z}$ there exist $u, u' \in E_\alpha, v \in E_\beta$ with

$$\alpha \circ \beta(n) + \alpha \circ \beta(m) - \alpha \circ \beta(n+m) =$$

$$\alpha(\beta(n)+\beta(m))+u-\alpha(\beta(n)+\beta(m)-v)=$$
$$\alpha(\beta(n)+\beta(m))+u-(\alpha(\beta(n)+\beta(m))+\alpha(-v)-u')=u-\alpha(-v)-u'.$$

We conclude that the expression $\alpha\circ\beta(n)+\alpha\circ\beta(m)-\alpha\circ\beta(n+m), n,m\in\mathbf{Z}$, takes its values in the finite set $\{u-\alpha(-v)-u' \mid u,u'\in E_\alpha,\ v\in E_\beta\}$. Hence, the map $\alpha\circ\beta$ and, with the same justification, also the map $\alpha'\circ\beta'$ are slopes.

Let $E_{\alpha,\alpha'}$ and $E_{\beta,\beta'}$ be finite sets such that we have $\alpha(n)-\alpha'(n)\in E_{\alpha,\alpha'}$ and $\beta(n)-\beta'(n)\in E_{\beta,\beta'}$ for $n\in\mathbf{Z}$. Hence, for $n\in\mathbf{Z}$ there exist $r\in E_{\alpha,\alpha'}, s\in E_{\beta,\beta'}$ and $u\in E_\alpha$ with

$$\alpha\circ\beta(n)-\alpha'\circ\beta'(n)=\alpha(\beta'(n)-s)-(\alpha(\beta'(n))+r)=$$
$$\alpha(\beta'(n))+\alpha(-s)-u-(\alpha(\beta'(n))+r)=\alpha(-s)-r-u.$$

We conclude that the expression $\alpha\circ\beta(n)-\alpha'\circ\beta'(n), n\in\mathbf{Z}$, takes its values in a finite set. Hence, the slopes $\alpha\circ\beta$ and $\alpha'\circ\beta'$ are equivalent. ■

Let $\mathbf{N}:=\{n\in\mathbf{Z}\mid n\geq 0\}$ be the set of natural numbers. We call a slope λ *positive*, if the set $\{\lambda(n)\mid n\in\mathbf{N},\lambda(n)\leq 0\}$ is finite, while the set $\{\lambda(n), n\in\mathbf{Z}\}$, is infinite. A real number a is *positive*, if its representing slopes are positive.

From this definition of positivity we obtain the ordering of the real numbers as usual in the following way. If a is positive, we say that $a>0$ and $0<a$ hold. The real number a is defined to be *less* then the real number b if there exists a positive real number t with $b=a+t$. If a is less then b, we say that $a<b$ holds.

More examples. We illustrate the definitions by examples before stating and verifying that the set $\mathbf{R}$ with the addition $+$, multiplication $\cdot$ and order relation $<$ satisfies all the axioms of the real numbers, i.e. of a complete totally ordered archimedean field. It is not necessary to understand all examples of this section at its first reading. Please feel free to jump to next section. Our choice of examples follows *engros* the historical appearence of different types of numbers.

A map $f:\mathbf{Z}\to\mathbf{Z}$ is called *odd* if for all $n\in\mathbf{Z}$ the property $f(-n)=-f(n)$ holds. An odd map $f:\mathbf{Z}\to\mathbf{Z}$ is determined by its restriction to $\mathbf{N}_+:=\{n\in\mathbf{Z}\mid n>0\}$. Let λ be an arbitrary slope. Then the map $\kappa:\mathbf{Z}\to\mathbf{Z}$ with $\kappa(0)=0$ defined by $\kappa(n):=\lambda(n), n>0$, and by $\kappa(n):=-\lambda(-n), n<0$, is an odd slope, which is equivalent to the slope λ. So every real number can be represented by an odd slope. In order to verify that an odd map $\gamma:\mathbf{Z}\to\mathbf{Z}$ is a slope, it suffices to check that the set $\{\gamma(n+m)-\gamma(n)-\gamma(m)\mid n,m\in\mathbf{N}_+\}$ is finite.

We will construct a slope that represents the number $\sqrt{2}$. Let $\rho:\mathbf{Z}\to\mathbf{Z}$ be the odd map defined by

$$\rho(n)=\min\{k\in\mathbf{N}\mid 2n^2\leq k^2\}, n\in\mathbf{N}_+$$

From the definition of $\rho(n)$ we have for $n\in\mathbf{N}_+$ the inequalities:

$$2n^2\leq\rho(n)^2, (\rho(n)-1)^2\leq 2n^2$$

and it follows $n \leq \rho(n) \leq 2n$. Hence

$$2n^2 \leq \rho(n)^2 \leq 2n^2 + 2\rho(n) - 1 \leq 2(n+1)^2$$

For $n, m \in \mathbf{N}_+$ we deduce

$$2nm \leq \rho(n)\rho(m) \leq 2(n+1)(m+1)$$

The map ρ is a slope, since for $n, m \in \mathbf{N}_+$ we estimate

$$x := (-\rho(n+m) + \rho(n) + \rho(m))(\rho(n+m) + \rho(n) + \rho(m)) =$$

$$-\rho(n+m)^2 + \rho(n)^2 + \rho(m)^2 + 2\rho(n)\rho(m)$$

by

$$-4n - 4m - 2 = -2(n+m+1)^2 + 2n^2 + 2m^2 + 4nm \leq x \leq$$

$$-2(n+m)^2 + 2(n+1)^2 + 2(m+1)^2 + 4(n+1)(m+1) = 8m + 8n + 8$$

and with $\rho(n+m)+\rho(n)+\rho(m) \geq n+m+1$, we conclude $|\rho(n+m)-\rho(n)-\rho(m)| \leq 8$. The equivalence class of ρ is a positive real number a satisfying $a^2 = 2$. Indeed, the number a^2 is represented by the composition $\rho \circ \rho$. We have for $n \in \mathbf{N}_+$ the inequalities

$$4n^2 \leq 2\rho(n)^2 \leq \rho(\rho(n))^2 \leq 2(\rho(n)+1)^2 \leq 4n^2 + 8n + 2 \leq 4(n+1)^2$$

showing $2n \leq \rho(\rho(n)) \leq 2n+2$, which means that the slopes $\rho \circ \rho$ and $\bar{2}$ are equivalent and represent the integer 2. Hence ρ represents the square root $\sqrt{2}$ of 2, which is the length of the diagonal of a unit square and can not be represented as a fraction $\frac{p}{q}$, see [F,V].

We will now construct a real number that is a root of the polynomial $p(x) := x^5 + x - 3$. Let the odd map $\alpha : \mathbf{Z} \to \mathbf{Z}$ be defined by

$$\alpha(n) := \min\{k \in \mathbf{N} \mid 3n^5 \leq k^5 + n^4 k\} = \min\{k \in \mathbf{N} \mid p(\frac{k}{n}) \geq 0\}, n \in \mathbf{N}_+$$

From the definition of $\alpha(n)$ we conclude

$$p(\frac{\alpha(n)-1}{n}) < 0 \leq p(\frac{\alpha(n)}{n})$$

From $p(2) > 0$, $p(1) = -1$ and the monotonicity of the map $r \in \mathbf{Q} \mapsto p(r) \in \mathbf{Q}$ we deduce

$$\frac{\alpha(n)-1}{n} \leq 2, \quad \frac{\alpha(n)}{n} \geq 1$$

We will show that the map α is a slope and represents the real root a of the equation $x^5 + x - 3 = 0$. This root a can not be represented by a compound radical expression after the work of Paolo Ruffini (1762-1822) and of Niels Henrik Abel (1802–1829), see [A,R,S].

First we show that α is a slope. For $n, m \in \mathbf{N}_+$ fixed let a_-, a_+ be the rational numbers

$$a_- := \max\{\frac{\alpha(n)-1}{n}, \frac{\alpha(m)-1}{m}, \frac{\alpha(m+n)-1}{m+n}\}$$

and

$$a_+ := \min\{\frac{\alpha(n)}{n}, \frac{\alpha(m)}{m}, \frac{\alpha(m+n)}{m+n}\}$$

We have $a_- \leq a_+$. From the monotonicity of p and the definition of α we deduce $p(a_-) \leq 0, p(a_+) \geq 0$. Let A be any rational number with $a_- \leq A \leq a_+$. From

$$\frac{\alpha(n)-1}{n} \leq a_- \leq A \leq a_+ \leq \frac{\alpha(n)}{n}$$

we deduce the inequality $|\alpha(n) - nA| \leq 1$. We also obtain in the same way the inequalities $|\alpha(m) - mA| \leq 1$ and $|\alpha(m+n) - (m+n)A| \leq 1$. From last three inequalities it follows $|\alpha(m+n) - \alpha(m) - \alpha(n)| \leq 3$, hence α is a slope.

Let a be the real number that is represented by the slope α. We show $p(a) = 0$ by showing that the slope $\alpha^{\circ 5} + \alpha - \bar{3}$ is bounded. Here we have used the notation $\alpha^{\circ e}$ for the e-th iterate, $e \in \mathbf{N}$, of α. It is not at all easy to handle directly iterates of slopes. The following estimate helps out and is proved by induction upon the exponent $e \in \mathbf{N}_+$

$$|n^{e-1}\alpha^{\circ e}(n) - \alpha(n)^e| \leq n^{e-1}(1 + |\alpha(1)| + S_\alpha)^{e-1}, n \in \mathbf{N}_+,$$

where the quantity

$$S_\alpha := \max\{|\alpha(u+v) - \alpha(u) - \alpha(v)|, u, v \in \mathbf{Z}\}$$

measures the non-additivity of the slope α. Note also $|\alpha(n)| \leq |n|(|\alpha(1)| + S_\alpha)$. It follows that the slope $\alpha^{\circ e}$ is equivalent to the odd slope defined by

$$n \in \mathbf{N}_+ \mapsto [\frac{\alpha(n)^e}{n^{e-1}}]_{\text{Gauss}}$$

So the slope $\alpha^{\circ 5} + \alpha - \bar{3}$ is equivalent to the odd slope ϵ defined by

$$n \in \mathbf{N}_+ \mapsto [\frac{\alpha(n)^5}{n^4}]_{\text{Gauss}} + \alpha(n) - 3n$$

The slope ϵ is bounded, since for $n \in \mathbf{N}_+$ we have with the monotonicity of p' the inequalities

$$0 \leq \epsilon(n) = np(\frac{\alpha(n)}{n}) \leq n(p(\frac{\alpha(n)-1}{n} + \frac{1}{n})) \leq$$

$$n(p(\frac{\alpha(n)-1}{n}) + \frac{p'(2)}{n}) \leq p'(2) = 81$$

If one does not want the use of the derivative p', one can with the binomial formula prove for $0 \leq x \leq x+h \leq 2,\ 0 < h < 1$, the inequality $p(x+h) - p(x) = (x+h)^5 + (x+h) - x^5 - x \leq 212h$ and obtain the bound

$$0 \leq \epsilon(n) \leq 212$$

. It follows the claim $p(a) = 0$. Here $[?]_{\text{Gauss}}$ is the Gaussian integral part bracket $[x]_{\text{Gauss}} := \max\{k \in \mathbf{Z} \mid k \leq x\}, x \in \mathbf{R}$.

Let $\beta : \mathbf{Z} \to \mathbf{Z}$ be the odd map with $\beta(n) := \#\{(p,q) \in \mathbf{Z} \times \mathbf{Z} \mid p^2 + q^2 \leq n\}, n > 0$. Unit squares in the plane with centers at the lattice points $(p,q) \in \mathbf{Z} \times \mathbf{Z}, p^2 + q^2 \leq n$, cover the disk of radius $\sqrt{n} - \frac{1}{2}\sqrt{2}$ and are contained in the disk of radius $\sqrt{n} + \frac{1}{2}\sqrt{2}$. Hence $|\beta(n) - n\pi| \leq 2\sqrt{2}\sqrt{n}, n \in \mathbf{N}$. It follows that the odd map $\bar{\beta}$ defined by $n \in N_+ \mapsto [\frac{\beta(n^2)}{n}]$ is a slope. The map β is not a slope since the quantity $s(n) := \beta(n) - \beta(n-1) - \beta(1)$ is not bounded. One has for instance $s(5^u) = 4u - 1, u \in \mathbf{N}_+$.

The slope $\bar{\beta}$ represents the area π of the unit disk in the plane, see Chap. 1 in *Anschauliche Geometrie* of David Hilbert & S. Cohn-Vossen [H-V]. Johann Heinrich Lambert (1728-1777) has proved in his communication of 1761 to the Academy in Berlin [Lam] that the number π is not a rational number and has conjectured that π is not root of a polynomial equation with integral coefficients, see [Le], [E-L]. Carl Louis Ferdinand von Lindemann [Li] has proved this conjecture in 1882, see [B].

The number e appeared in the sixteenth century, when it was noticed that the expression $(1+\frac{1}{n})^n$ for compound interest increases with n to a certain value $2.7182818\cdots$, see the book "**e** *The Story of a Number*" by Eli Maor [M]. The number e became of central importance in Mathematics since its interpretations in Geometry and Analysis by Grégoire de Saint-Vincent (1584-1667). It is not obvious to define the number e with a slope. We use the solution to a problem, see [D], of Jakob Steiner (1796-1863) and define for $n \in \mathbf{N}, n > 0$, the integer $\epsilon(n)$ to be the natural number $k, k > 0$, such that the expression $(\frac{k}{n})^{\frac{n}{k}}$ takes its maximal value. The corresponding odd function $\epsilon : \mathbf{Z} \to \mathbf{Z}$ is a slope representing the number e.

The classical construction of the system of real numbers is based on Dedekind cuts or on Cauchy sequences $(r_n)_{n \in N}$ of rational numbers. The present construction by slopes is related to the classical ones as follows: To a slope λ corresponds a Dedekind cut (A, B) by setting

$$A := \{\frac{p}{q} \in \mathbf{Q} \mid \bar{p} \leq \lambda \circ \bar{q}\}$$

and

$$B := \{\frac{p}{q} \in \mathbf{Q} \mid \lambda \circ \bar{q} \leq \bar{p}\}$$

and also a Cauchy sequence $(r_n)_{n \in N}$ by setting $r_n := \frac{\lambda(n+1)}{n+1}$.

Well adjusted slopes.

We call a slope λ *well adjusted* if it is odd and satisfies the inequalities

$$-1 \leq \lambda(m+n) - \lambda(m) - \lambda(n) \leq 1, n, m \in \mathbf{Z}$$

One can say that a well adjusted slope need not be an additive map from $\mathbf{Z}$ to $\mathbf{Z}$, but deviates as little as possible from being additive. Each slope is equivalent to a well adjusted slope, as shows the concentration Lemma below. So in particular, a real number can be represented by a well adjusted slope.

For integers $p, q, q \neq 0$, the result of optimal euclidean division of p by q will be denoted by $p : q$. The *optimal euclidean division* is the integer $r := p\!:\!q \in \mathbf{Z}$ that satisfies the inequalities $2p - |q| \leq 2qr < 2p + |q|$, where $|q| := \max\{q, -q\}$ is the absolute value of q. For instance $4:7 = 1$ but $3:7 = 0$. If $p/q, p, q \in \mathbf{Z}, q \neq 0$, denotes the fraction, then we have $|p/q - p:q| \leq 1/2$. For the optimal euclidean division, we have

Lemma 2 *Let $q \in \mathbf{N}_+$ and $a, b, c \in \mathbf{Z}$ be such that $-q \leq a - b - c \leq q$. Then we have*

$$-1 \leq a:3q - b:3q - c:3q \leq 1$$

Proof. The integer

$$a:3q - b:3q - c:3q$$

differs from the integer 0 by at most

$$1/2 + 1/2 + 1/2 + |a/3q - b/3q - c/3q| \leq 3/2 + 1/3 = 11/6$$

Hence we have

$$-1 \leq a:3q - b:3q - c:3q \leq 1$$

since $11/6 < 2$. ■

Lemma 3 *Let $n, m \in \mathbf{N}_+$ and $c \in \mathbf{Z}$. Then we have*

$$-1 \leq c:m(n+m) + c:n(n+m) - c:nm \leq 1.$$

Proof. The integer

$$c:m(n+m) - c:n(n+m) - c:nm$$

differs from the integer

$$0 = c/m(n+m) + c/n(n+m) - c/nm$$

by at most $1/2 + 1/2 + 1/2 = 3/2$, hence

$$-1 \leq c:m(n+m) - c:n(n+m) - c:nm \leq 1$$

since $3/2 < 2$. ■

Lemma 4 (Concentration Lemma) *Let λ be a slope. Let $s \in \mathbf{N}_+$ be such that for all $n, m \in \mathbf{Z}$ we have $-s \leq \lambda(m+n) - \lambda(m) - \lambda(n) \leq s$. Let $\lambda' : \mathbf{Z} \to \mathbf{Z}$ be defined by $\lambda'(n) := \lambda(3sn):3s, n \in \mathbf{Z}$. Then the map λ' is a well adjusted slope, which is equivalent to the slope λ.*

Proof. By induction on $t \in \mathbf{N}_+$, we prove $-s(t-1) \leq \lambda(tn) - t\lambda(n) \leq s(t-1)$. Setting $t = 3s$ we get

$$-s(3s-1) \leq \lambda(3sn) - 3s\lambda(n) \leq s(3s-1)$$

and after dividing by $3s$ and using $|\lambda(3sn)/3s - \lambda'(n)| \leq 1/2$ we get

$$-s + (1/3 - 1/2) \leq \lambda'(n) - \lambda(n) \leq s - (1/3 - 1/2)$$

which shows for the integers $s, \lambda'(n), \lambda(n)$ the inequality

$$-s \leq \lambda'(n) - \lambda(n) \leq s$$

and hence the equivalence of the slopes λ and λ'. With lemma 2 we deduce from

$$-s \leq \lambda(3sn + 3sm) - \lambda(3sn) - \lambda(3sm) \leq s$$

the inequality

$$-1 \leq \lambda'(n+m) - \lambda'(n) - \lambda'(m) \leq 1$$

, hence the slope λ' is well adjusted. ∎

A well adjusted slope λ has the following properties:
- $|\lambda(n+1) - \lambda(n)| \leq |\lambda(1)| + 1$,
- if for some $k \in \mathbf{N}_+$ we have $\lambda(k) > 1$ (or $\lambda(k) < -1$), then we have for any $n \in \mathbf{N}_+$ the inequality $\lambda(n) \geq -1 + n : k$ (or $\lambda(n) \leq +1 - n : k$),
- if for some $k \in \mathbf{Z}$ we have $\lambda(k) > 1$, then for $v \in \mathbf{Z}$ the set $\{n \in \mathbf{Z} \mid \lambda(n) = v\}$ is finite and has fewer then $k+1$ elements,
- if for some $k \in \mathbf{Z}$ we have $\lambda(k) > 1$, then for any $v \in \mathbf{Z}$, there exists $n \in \mathbf{Z}$ with $|v - \lambda(n)| \leq |\lambda(1)| + 1$,
- the real number x represented by λ satisfies $x > 0$ if and only if there exists $a \in \mathbf{N}$ with $\lambda(a) > 1$,
- if λ represents a real number $x > 0$, then $\lambda(n) \geq 0$ for all $n \in \mathbf{N}_+$,
- let y be a real number represented by a well adjusted slope κ. We have $x > y$ if and only if there exists $n \in \mathbf{N}_+$ with $\lambda(n) > 2 + \kappa(n)$.

The following Lemma controls the finite differences $\lambda(n+k) - \lambda(n)$ of a slope.

Lemma 5 (Finite Differences Lemma) *Let λ be a slope. If λ takes infinitely many values, then there exist $b, B \in \mathbf{N}_+$ such that the following inequalities hold:*

$$|\lambda(n+k) - \lambda(n)| \leq kb, n \in \mathbf{Z}, k \in \mathbf{N},$$

$$|\lambda(n+kB) - \lambda(n)| \geq k, n \in \mathbf{Z}, k \in \mathbf{N}.$$

In particular, the slope λ takes each value at most $2B-1$ times.

Proof. Let s_λ be an upper bound for $\lambda(n+m) - \lambda(n) - \lambda(m)$, $n, m \in \mathbf{N}$. We have

$$|\lambda(n+k) - (\lambda(n) + \lambda(1) + \cdots + k - \text{times} \cdots + \lambda(1))| \leq ks_\lambda$$

hence, putting $b = s_\lambda + |\lambda(1)|$, wet obtain the first inequality of the lemma. Let $B \in \mathbf{N}_+$ be such that $|\lambda(B)| > s_\lambda$. We have

$$|\lambda(n + kB) - (\lambda(n) + \lambda(B) + \cdots + k - \text{times} \cdots + \lambda(B))| \leq ks_\lambda$$

hence, since $|\lambda(B)| \geq s_\lambda + 1$, we obtain the second inequality. ■

There are many well known examples of well adjusted slopes. For a natural number $a > 0$ let its length $L(a)$ be 1 less then the number of digits that appear in the decimal notation of a. The odd function $\lambda : \mathbf{Z} \to \mathbf{Z}$ defined by $\lambda(n) := L(a^n)$, $n > 0$, is a well adjusted slope that represents the real number $\text{Log}_{10}(a)$. The importance of the Log_{10} function lies in the property:

$$\text{Log}_{10}(ab) = \text{Log}_{10}(a) + \text{Log}_{10}(b)$$

The length function L satisfies this property with error at most 1. We have:

$$-1 \leq L(ab) - L(a) - L(b) \leq 1$$

hence L appears as a well adjusted map that transform multiplication of natural numbers in addition of natural numbers. We get finally the following definition for $\text{Log}_{10}(x)$ of a real number x, $x \geq 1$:

$$\text{Log}_{10}(x) := [n \in \mathbf{N} \mapsto L([x^n]_{Gauss})]_{ec}$$

where $[?]_{ec}$ means equivance class of ?. For $0 < x \leq 1$ we define $\text{Log}_{10}(x) := -\text{Log}_{10}(1/x)$.

The axioms.

We now state, partially in abbreviated form, the axioms for a complete totally ordered field, that are satisfied by the quadruple $(\mathbf{R}, +, \cdot, <)$. The presentation of the axioms is slightly redundant.

1. The pair $(\mathbf{R}, +)$ is an abelian group.
2. The triple $(\mathbf{R}, +, \cdot)$ is a field.
3. The quadruple $(\mathbf{R}, +, \cdot, <)$ is an archimedean, complete, totally ordered field.

   Complete ordered field, i.e.

   - for any non-empty subset T bounded from above in $\mathbf{R}$ there exists a least upper bound in $\mathbf{R}$ called the *supremum* of T. It will be denoted by Sup T.

   Archimedian ordered field, i.e.

   - for $a \in \mathbf{R}, a > 0$ and $A \in \mathbf{R}$ there exists $N \in \mathbf{N}$ such that $Na > A$.

We now begin the verification of the axioms for the system of real numbers, that we have introduced above. We leave out those verifications, that are straightforward and can be done without using well adjusted slopes as representatives.

The addition $+$ of integers makes Z into an abelian group $(Z, +)$. It follows easily that $(\mathbf{R}, +)$ is also an abelian group.

The triple $(\mathbf{R}, +, \cdot)$ is a field, i.e. verifies all the field axioms. Multiplication is associative since the composition of maps is. Only commutativity and the existence of inverses need extra care. For two slopes α, β we have the estimates

$$n\alpha(\beta(n)) = \alpha(n\beta(n)) + E_1 = \alpha(\beta(n)n) + E_1 = \beta(n)\alpha(n) + E_2 + E_1$$

with $|E_1| \leq |n|S_\alpha$ and $|E_2| \leq |\beta(n)|S_\alpha \leq |n|(|\beta(1) + S_\beta)S_\alpha$. It follows

$$|\alpha \circ \beta(n) - \beta \circ \alpha(n)| \leq S_\alpha(1 + |\beta(1)| + S_\beta) + S_\beta(1 + |\alpha(1)| + S_\alpha)$$

showing that the slopes $\alpha \circ \beta$ and $\beta \circ \alpha$ are equivalent, hence the multiplication is commutative.

Let 1 be the real number represented by the identity map $\mathrm{Id}_\mathbf{Z} : \mathbf{Z} \to \mathbf{Z}$. Clearly we have for a real number x the properties $1x = x1 = x$, which makes 1 into the unit element of the multiplication $\cdot$ of $\mathbf{R}$.

We now construct a right inverse for $x \in \mathbf{R}, x \neq 0$, i.e. an element $y \in \mathbf{R}$ satisfying $xy = 1$. Let α be a well adjusted representing slope for x. It follows, that for each $v \in \mathbf{Z}$ the set $A_v := \{n \in \mathbf{Z} \mid |v - \alpha(n_v)| \leq |\alpha(1)| + 1\}$ is not empty and finite. Let $n_v \in \mathbf{Z}$ be the smallest element in A_v. We define a map $\beta : \mathbf{Z} \to \mathbf{Z}$ by $\beta(v) := n_v$.

We claim that the map β is a slope. Indeed, for $v, w \in \mathbf{Z}$ we have

$$|\alpha(\beta(v + w) - \beta(v) - \beta(w))| = |\alpha(n_{v+w} - n_v - n_w)| \leq$$

$$|(v + w) - v - w| + 2 + 3(|\alpha(1)| + 1) = 3|\alpha(1)| + 5.$$

Since α takes each value only finitely many times, we conclude that the set $\{\beta(v + w) - \beta(v) - \beta(w) \mid v, w \in \mathbf{Z}\}$ is finite.

For $v \in \mathbf{Z}$ we have $\alpha \circ \beta(v) = \alpha(n_v)$, so the slopes $\alpha \circ \beta$ and $\mathrm{Id}_\mathbf{Z}$ are equivalent, since $|v - \alpha(n_v)| \leq |\alpha(1)| + 1$ holds. It follows that $xy = 1$.

The pair $(\mathbf{R}, <)$ is an order relation. First we prove that the relation $<$ is total. Let x, y be real numbers represented by the slopes α and β. We consider the slope $\delta := \alpha - \beta$, which represents the number $x - y$. Let δ' be the well adjusted slope equivalent to δ given by the concentration lemma. If $x \neq y$ the slope δ' is not bounded and we have for some $n \in \mathbf{N}$ either $\delta'(n) > 1$, or $\delta'(n) < -1$. In the first case we have $\delta'(kn) > k$ for $k \in \mathbf{N}_+$, hence $x > y$. In the second case we can show $x < y$. If $x = y$ the slopes δ and δ' are bounded and hence the case $x = y$ excludes both $x < y$ and $x > y$. The cases $x < y$ and $x > y$ exclude each other. It remains only to prove transitivity. Let x, y, z be real numbers with $x > y$ and $y > z$, which are represented by the slopes α, β and γ. Let δ_1, δ_2 be well adjusted slopes equivalent to the slopes $\alpha - \beta$ and $\beta - \gamma$. So for some $n \in \mathbf{N}_+$ and $m \in \mathbf{N}_+$ we have $\delta_1(n) > 1$ and $\delta_2(m) > 1$. From $\delta_1(nm), \delta_2(nm) > 1$ it follows that $(\delta_1 + \delta_2)(nm) > 3$. The slope $\delta_{12} := \delta_1 + \delta_2$ satisfies $-2 \leq \delta_{12}(n + m) - \delta_{12}(n) - \delta_{12}(m) \leq 2$ all $n, m \in \mathbf{Z}$ and by induction on $k \in \mathbf{N}_+$ we prove $\delta_{12}(knm) \geq 2k$. Hence we have $x > z$.

The quadruple $(\mathbf{R}, +, \cdot, <)$ is an ordered field. Let x, y be reals satisfying $x < y$ and let t be real. We represent x, y, t by the well adjusted slopes α, β and τ. Since $x < y$ there exists $b \in \mathbf{N}$ with $\beta(bn) > \alpha(bn) + n, n \in \mathbf{N}_+$. Hence, $\beta(bn) + \tau(bn) < \alpha(bn) + \tau(bn) + n, n \in \mathbf{N}_+$, showing the monotonicity property for translations $x + t < y + t$. If $t > 0$, for some $d \in \mathbf{N}$ we have $\tau(dn) > n, n \in \mathbf{N}_+$, hence $\beta(dbn) > \alpha(dbn) + dn$ by the definition of b and

$$\tau(\beta(bdn)) > \tau(\alpha(dbn) + dn)) + \tau(\beta(dbn) - \alpha(dbn) - dn) - 1 >$$

$$\tau(\alpha(dbn)) + \tau(dn) - 1 + \tau(\beta(dbn) - \alpha(dbn) - dn) - 1$$

Now, $\tau(\beta(dbn) - \alpha(dbn) - dn) \geq 0$ since τ being well-adjusted, representing $t > 0$ and $\beta(dbn) - \alpha(dbn) - dn \in \mathbf{N}_+$, and moreover $\tau(dn) > n$ by the definition of d. Putting together yields $\tau(\beta(bdn)) > \tau(\alpha(dbn) + n - 1$ showing the monotonicity property for stretchings $tx < ty$.

We now prove the archimedean property. Let $a \in \mathbf{R}$ with $a > 0$ and $A \in \mathbf{R}$ be given. We construct $N \in \mathbf{N}$ such that $Na > A$, as follows. We represent a and A by well adjusted slopes λ and Λ. Since $a > 0$ holds, we may choose $n \in \mathbf{N}_+$ with $\lambda(n) > 1$. Then $\lambda(2n) > 2$. Define $N := 1 + \max\{\Lambda(2n), 0\}$. Let κ be the well adjusted slope equivalent to the slope $N\lambda$. We have $\kappa(2n) > N\lambda(2n) - N > 2 + \Lambda(2n)$. Hence $Na > A$.

Finally, we will establish the completeness. Let D be a non-empty subset in $\mathbf{R}$ bounded from above by $m \in \mathbf{R}$. So for $x \in D$ we have the inequality $x \leq m$. Let Δ be a set of well adjusted slopes representing the real numbers in the set D. Let μ be a well adjusted slope representing m. For every $n \in \mathbf{N}$ and every $\delta \in \Delta$ we have $\delta(n) < \mu(n) + 2$. It follows that for $n \in \mathbf{N}_+$ the non-empty set $\{\delta(n) \mid \delta \in \Delta\}$ is bounded from above by $\mu(n) + 2$. Let $\sigma : \mathbf{Z} \to \mathbf{Z}$ be the odd map defined by

$$\sigma(n) := \max\{\delta(n) \mid \delta \in \Delta\}.$$

We claim that the map σ is a slope. Indeed, for $u \in \mathbf{N}_+$ let $\delta_u \in \Delta$ be a slope, which attains at u the value $\max\{\delta(u) \mid \delta \in \Delta\}$. So, we have $\delta_u(u) = \sigma(u)$. For $p, N \in \mathbf{N}_+$ put $q := pN$. We compare δ_p, δ_q at p and q as follows. We have

$$\delta_q(q) : N \leq \delta_p(p) + 1$$

since $|\delta_q(q) : N - \delta_q(p)| \leq 1$ and $\delta_q(p) \leq \delta_p(p)$. We also have

$$N\delta_p(p) \leq \delta_p(q) + N \leq \delta_q(q) + N.$$

We conclude that for all $p, N \in \mathbf{N}_+$

$$|\delta_p(p) - \delta_{pN}(pN) : N| \leq 1.$$

Hence for $n, m \in \mathbf{N}_+$, where we put $c := \delta_{nm(n+m)}(nm(n+m))$, the following inequalities hold:

$$|\sigma(n) - c : m(n+m)| \leq 1,$$
$$|\sigma(m) - c : n(n+m)| \leq 1,$$
$$|\sigma(n+m) - c : nm| \leq 1.$$

For instance, the first inequality is obtained with $p = n, N = m(n+m), q = Np$ and by comparing δ_p and δ_q at the point p. From

$$|c : nm - c : m(n+m) - c : n(n+m)| \leq 1$$

it follows that for all $n, m \in \mathbf{N}_+$ we have

$$|\sigma(n+m) - \sigma(n) - \sigma(m)| \leq 1 + 3 = 4,$$

which proves our claim.

Let s be the real number represented by the slope σ. For all $x \in D$ we have the inequality $x \leq s$, since for a slope $\delta \in \Delta$ representing x the inequalities

$$\delta(n) \leq \delta_n(n) = \sigma(n), n \in \mathbf{N}_+,$$

hold. So $s \in \mathbf{R}$ is an upper bound for D.

In order to prove that s is the least upper bound of D, we show that no $t \in \mathbf{R}$ with $t < s$ is an upper bound of D. Indeed, let τ be a well adjusted slope for $t \in \mathbf{R}$ with $t < s$. There exists $n \in \mathbf{N}_+$ with $\tau(n) < \sigma(n) - 2$. Let x in D be represented by δ_n. We have $\delta_n(n) > \tau(n) + 2$, hence $x > t$ and t is not an upper bound for D.

Remarks: The above construction of the field of real numbers $(\mathbf{R}, +, \cdot)$ has as its starting point the additive group $(\mathbf{Z}, +)$. It is well known that the order relation $<$ on $\mathbf{R}$ is encoded in the field structure of $\mathbf{R}$, namely for $x, y \in \mathbf{R}$ we have $x < y$ if and only if $y - x = t^2$ holds for some $t \in \mathbf{R} \setminus \{0\}$. So we see that the real ordered field $(\mathbf{R}, +, \cdot, >)$ is constructed directly out of the additive group of integers.

The group $\mathbf{R}/\mathbf{Z}$ appears as second bounded cohomology group $H_b^2(\mathbf{Z}, \mathbf{Z})$ with coefficients $\mathbf{Z}$ of the group $\mathbf{Z}$. Bounded cohomology is defined by Michael Gromov in the seminal paper [G]. An element $\theta \in H_b^2(\mathbf{Z}, \mathbf{Z})$ is by definition the class modulo the boundary df of a bounded 1-cochain $f : \mathbf{Z} \to \mathbf{Z}$ of a bounded 2-cocycle on $\mathbf{Z}$ with values in $\mathbf{Z}$. A bounded 2-cocycle on $\mathbf{Z}$ with values in $\mathbf{Z}$ is a bounded map $\theta : \mathbf{Z} \times \mathbf{Z} \to \mathbf{Z}$ that satisfies $d\theta = 0$. For example a real number a defines a bounded 2-cocycle $\theta_a(n, m) := [(n+m)a] - [na] - [ma]$. The map $a \in \mathbf{R} \mapsto \theta_a$ induces an isomorphism from $\mathbf{R}/\mathbf{Z}$ to $H_b^2(\mathbf{Z}, \mathbf{Z})$. We recall the formulae for differentials in group cohomology: $d\theta(u, v, w) := \theta(u, v) - \theta(u+v, w) + \theta(u, v+w) - \theta(v, w)$ and $df(n, m) := f(n) - f(n+m) + f(m)$. Our construction of the reals defines the additive group $(\mathbf{R}, +)$ as the quotient $\mathbf{R} := C_{db}^1(\mathbf{Z}, \mathbf{Z})/C_b^1(\mathbf{Z}, \mathbf{Z})$, with

$$C_{db}^1(\mathbf{Z}, \mathbf{Z}) := \{1-\textit{cochains on } \mathbf{Z} \textit{ with values in } \mathbf{Z} \textit{ and with bounded differential}\},$$

$$C^1_b(\mathbf{Z},\mathbf{Z}) := \{\mathit{bounded}\, 1\!-\!\mathit{cochains\ on}\, \mathbf{Z}\, \mathit{with\ values\ in}\, \mathbf{Z}\}.$$

The composition of maps in $C^1_{db}(\mathbf{Z},\mathbf{Z}) = \{\mathit{slopes}\}$ induces the multiplication in $\mathbf{R}$.

The encoding of the order relation on $\mathbf{R}$ in the field structure has far reaching consequences. For instance, the fact that a line incidence preserving bijection of the real projective plane is a projective transformation, is such a consequence.

The first rigorous definitions for real numbers were published independently in 1872 by G. Cantor, E. Heine, Ch. Méray, and R. Dedekind. The rigorous definition of convergence of a sequence of numbers was given by d'Alembert in 1765 and by Cauchy in 1821 without having at the time a rigorous definition of real numbers. An exposition of the construction of real numbers is given in the book "Grundlagen der Analysis" of Edmund Landau [Lan]. We recommend reading the book "Analysis by Its History" by E. Hairer and G. Wanner [H-W].

Our interest for finding an alternative definition of real numbers came from the experience that it was very difficult to explain these matters to first year university students. Especially, we wanted not to use words like *as small as you wish, epsilons, limit, ...* and to construct first the objects, i.e the real numbers, as simply as possible. The first hint, that such a construction must exist, was given by a Theorem of William Thurston [T] about the third cohomology group with integral coefficients of the classifying space of Haefliger foliations. The Theorem says that this group surjects by the Godbillon-Vey invariant, see [G-V], with finite kernel to $\mathbf{R}$. It is still an open question if this kernel is trivial or not. Having trivial kernel is equivalent to following assertion: Every smooth codimension 1 foliation on the 3-sphere S^3 with trivial Godbillon-Vey invariant extends to a smooth codimension 1 foliation on the 4-ball D^4. It was a good surprise for me, and happy for the students, that the real numbers can be constructed naturally and directly from the integers in a way that was much simpler as the above hint suggests.

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